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# Understanding Geometric Pattern and its Geometry Part 8 – Designing patterns with alternative tessellations in decagonal geometry

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## Abstract

*In this paper, we discuss an alternative approach to the construction of decagonal geometric patterns. We show how one can construct geometric structures for a pattern using alternative tessellations. For the sake of simplicity, we deal here with decagonal geometry and one type of pattern only. However, most of our discussion can be continued with some changes for geometric patterns in other geometries (e.g., octagonal or dodecagonal).*

## Introduction

In some publications related to geometric patterns in decagonal geometry, we deal with tessellations of regular pentagons, trapeziums derived from these pentagons, regular decagons, etc. In such cases, large elements of a pattern are located on the edges of tessellation tiles. This approach is very convenient when creating the geometric structure of a pattern. However, it is not natural. Most craftsmen, while dealing with a geometric pattern, prefer to deal with large pattern shapes frequently decorated with an extra motif. Thus it could be worth exploring tessellations following the craftsmen approach.

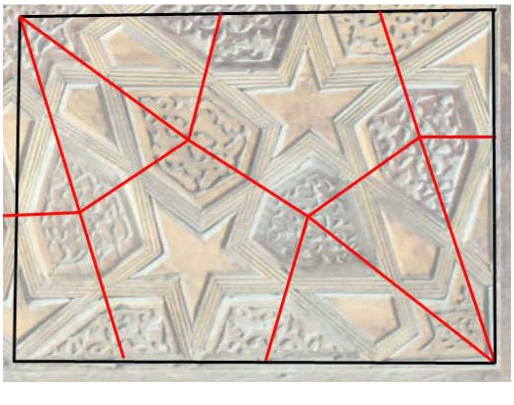


Fig. 1. Geometric pattern and its tessellation

The central part of each tessellation tile contains a minor element without any decoration. The large decorated petals are located on the edges of the tessellation tiles.

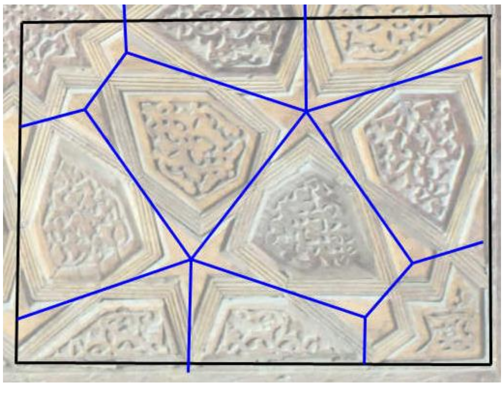


Fig. 2. An alternative tessellation

In this drawing, we deal with a tessellation where the central part of each tessellation tile contains a large decorated pattern shape.

[1] In 2018, New York Tech announced that it will no longer enroll students at its Abu Dhabi campus.

The above example, although very simple, gives an idea about the differences between these two types of tessellations. We will analyze a more complex example to familiarize ourselves better with this idea. Both examples were taken from the Kukeldash Madrasah main gate [2].

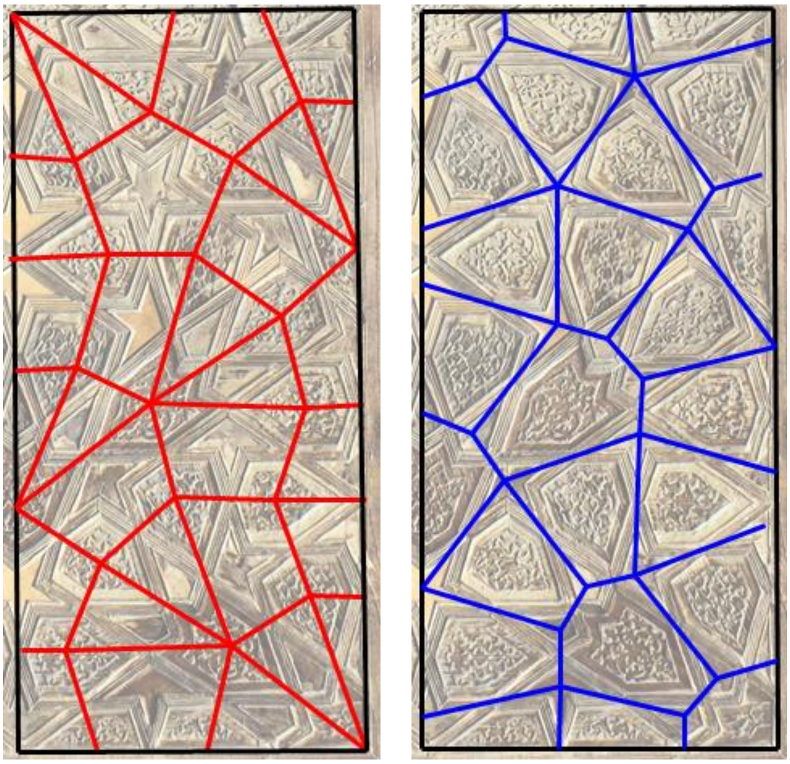


Fig. 3 and 4. Fragment of the main doors from Kukeldash Madrasah in Bukhara

In the left drawing, we see a pattern and a standard tessellation. The right drawing shows the same pattern with an alternative tessellation. A specific feature of it are chains with three or more large tiles.

None of the two tessellations shows all possible tessellation tiles that occur in decagonal geometric patterns. But we have some idea of what we may find. In one of the next sections, we will discuss the properties of tessellation tiles and their pattern shapes.

In this paper, we deal exclusively with tessellations and patterns in decagonal geometry, but we can continue the same, or very similar, investigations for patterns and their tessellations in other types of geometry. Moreover, we will limit our examples to the group of decagonal patterns with angles of 144 and 36 degrees between pattern edges (the so-called Kukeldash Madrasah style, see [5] ). The next figure will show how these two tessellations are related. This image will allow us to draw some conclusions and make some definitions.

DEFINITIONS

● The first type of tessellation we will call a *primary tessellation* or *D.a tessellation. D* stands here for decagonal.

● The second tessellation we will call an *alternative tessellation* or *D.b tessellation*. Consequently, tiles used in *D.a* tessellation we will call *D.a tiles*, and tiles used in *D.b* tessellations *D.b tiles*.

[2] Kukeldash Madrasah is one of the most iconic madrasah in Bukhara. It was built around 1568-69. The main gate to it is an original wooden construction, one of the oldest in Central Asia. Note, in Tashkent there is also another Kukeldash Madrasah. However, it is out of our interest.

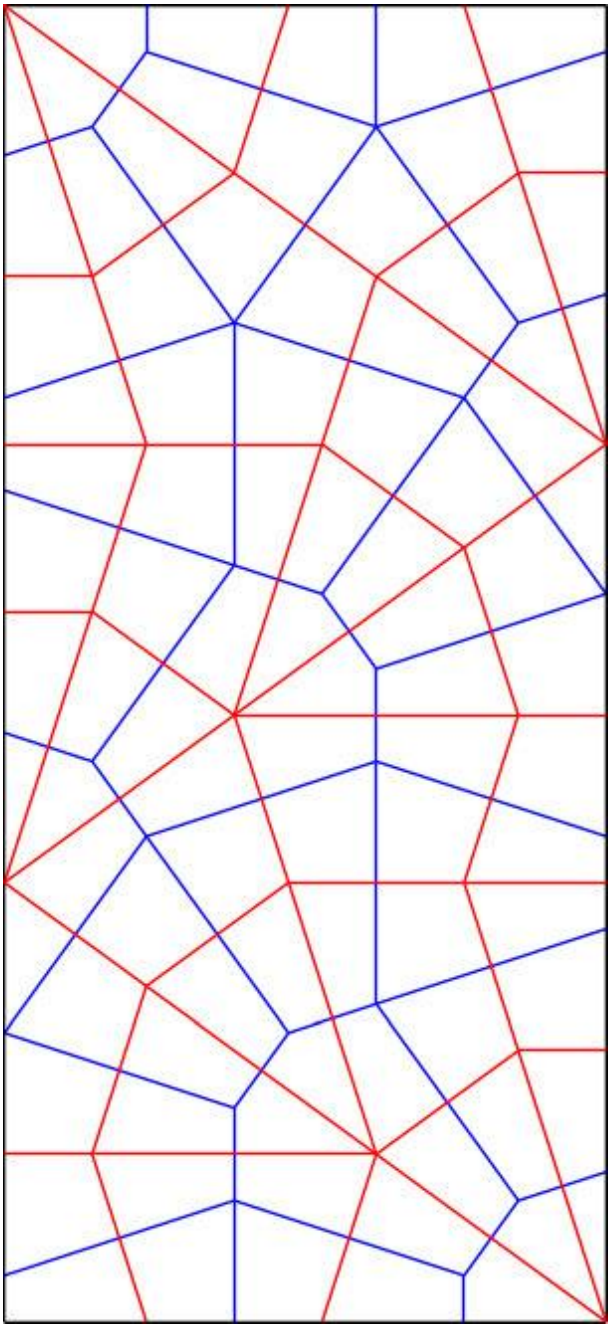


**Fig. 5 The primary and alternative tessellations for the Kukeldash Madrasah main doors pattern**

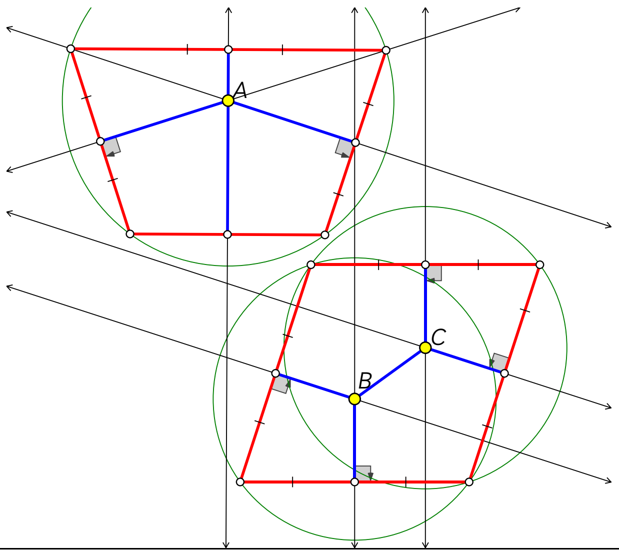
The drawing shows the relation between both tessellations. Each edge of one tessellation is
(1) perpendicular to an edge of the other tessellation, and
(2) their intersection point is the midpoint of both crossing edges.

**DEFINITION**

Tessellations fulfilling properties (1) and (2) we will call *orthogonal tessellations*.

Provided that one of these tessellations is given, it is easy to create the other tessellation. For each tessellation tile, we find the midpoints of its edges and draw through them perpendicular lines. This way, we get one or more intersection points. For some polygons, we have only one point that we call *circumcenter*. However, for some other polygons, we get only *partial circumcenters* (points B and C in the drawing below).

Fig. 5a




## Review of polygons and pattern shapes for primary tessellations

In most of the *D.a* tessellations for Kukeldash Madrasah style patterns, we deal with three or four polygons and four pattern shapes only. We know most of them from [5] The next series of drawings serve as a reminder. For further consideration, these shapes will be insignificant. They will be located on the edges of *D.b* tessellation tiles.

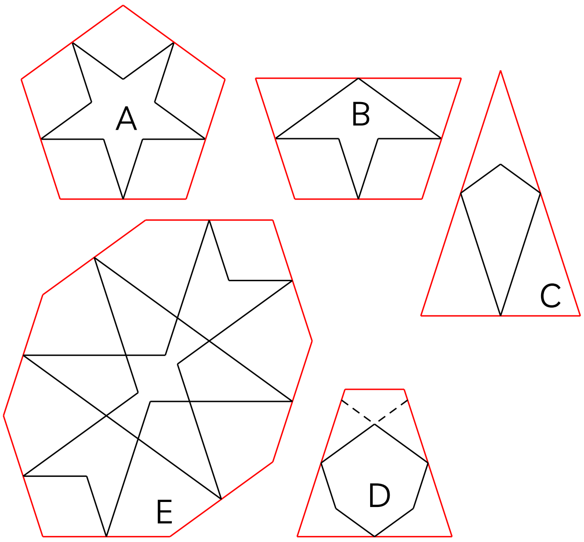



**Fig. 6. Tessellation tiles and pattern shapes for primary tessellations Kukeldash Madrasah style**

Elements A, B, and C are the most frequently used. Element C can be considered as a 1/10 of a regular decagon. Thus we do not show a complete regular decagon with its pattern.

Element E occurs very rarely, and we can find very few examples employing it.

Element D occurs more frequently than E, and it can have some extensions marked here with dashed lines.

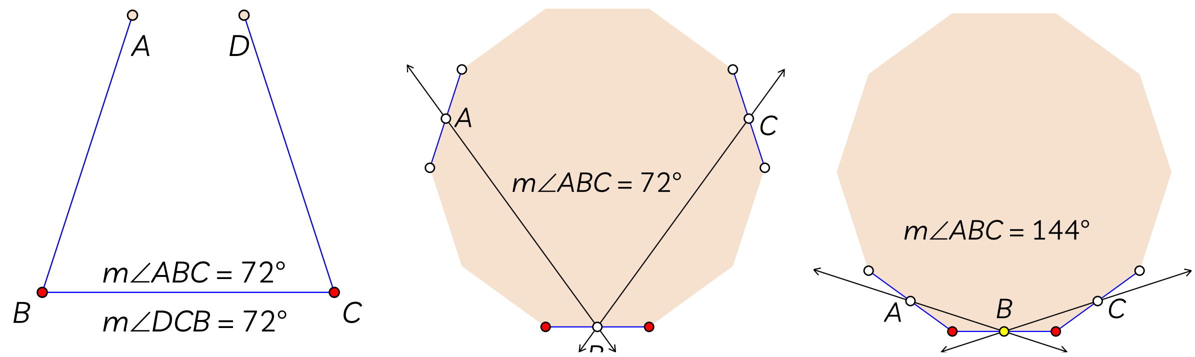
All elements presented here we have seen in some other papers in the series 'Understanding geometric pattern...' (see [4] [5] [6] ).

## Review of *D.b* tiles and pattern shapes for alternative tessellations

The groups of tessellation tiles and pattern shapes in alternative tessellations are significantly larger and contain several interesting elements. Each of these elements can be created from a regular decagon. This feature is especially interesting for designing them using Geometer's Sketchpad. For each of them, we can create a GSP tool based on two points that are vertices of a regular decagon. The following drawings will show some of the most frequently used polygons and their pattern shapes. The red points can be used to create an appropriate GSP tool (see [6] ).

It could be convenient before starting all constructions to prepare some tools for all future designs.

Fig. 7. Basic tools, *D.b* polygons, and pattern shapes




**Trapezium with three edges equal length** and two angles 72 degrees.

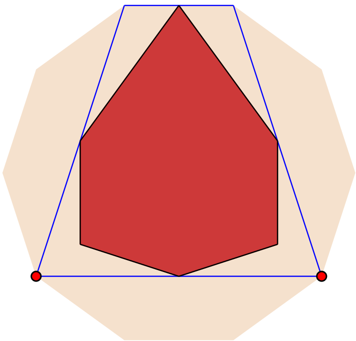
**Angle 72 degrees** is used in all following geometric shapes on short edges of tessellation tiles.

**Angle 144 degrees** is used on long edges of tessellation tiles.

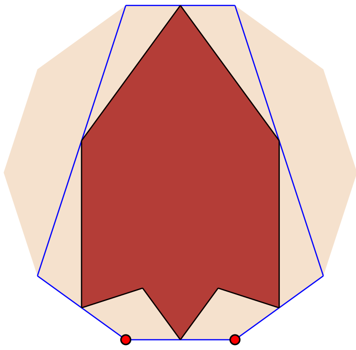
*GSP – a tool with points ABCD and blue lines is often needed to add a pattern shape somewhere in the middle of a design.*

*A GSP tool for it will be essential for all constructions in this paper.*

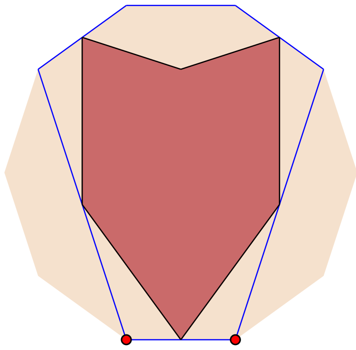
This is another important *GSP* tool for many constructions presented in this paper.



**Petal 1** – the most frequent shape used in all Kukeldash Madrasah style designs.



**Petal 2** is much longer than petal 1. We may need it occasionally to fill a gap in a design.



**Petal 3** is a variation of petal 2

## Comments

Each of these drawings uses the edges of a regular decagon and, in most cases, its medium chords. Thus we have:

**Petal 1** – is the only shape created with three medium chords.

**Petals 2, 3, 4**, and **5** use two medium chords.

**Petals 6, 7, 8**, and **8a** use one chord only.

It is possible to create more petals using one chord only. However, they are rare, and we ignore them here.

In some rare examples, we can find rather unusual shapes. We do not show them here.

The star shape uses short edges of a decagon only.

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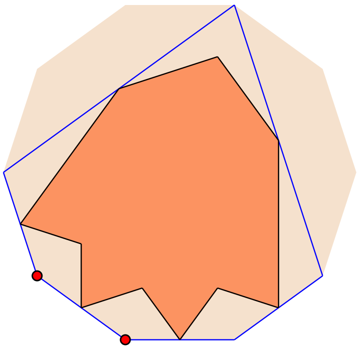
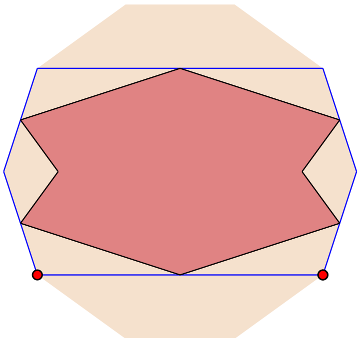
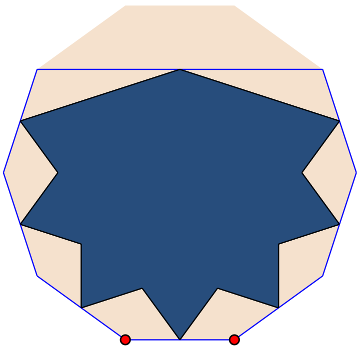


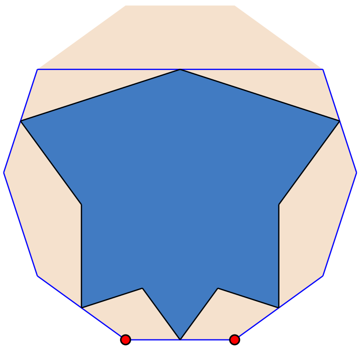
**Petal 4** – one of the most interesting shapes in decagonal patterns. Uses four edges of a decagon.



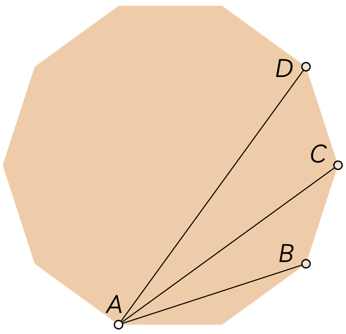
**Petal 5** – another variation of petal 2.



**Petal 6** – a shape using seven edges of a decagon and only one chord of the decagon.

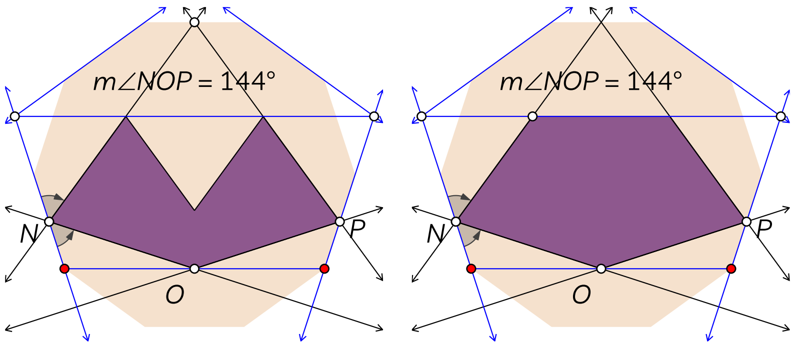


**Petal 7** – a variation of petal 6. It occurs very rarely and mostly in Maghrebi designs.



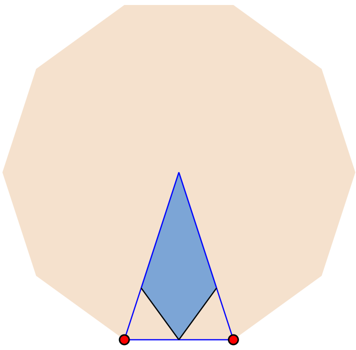

**Chords in a regular decagon**

AC is the medium chord.

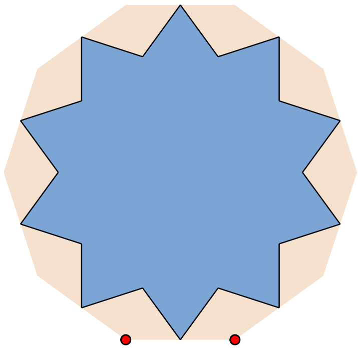



**Petal 8** – a very convenient pattern shape sometimes used in Maghrebi designs. This is the same shape that we often see in classic Ottoman designs. However, there it is always attached to petal 1.

**Petal 8a** – is a simplified form of petal 8.



**Petal 9** – single triangle occurs mostly in constructions of a simple decagonal star.

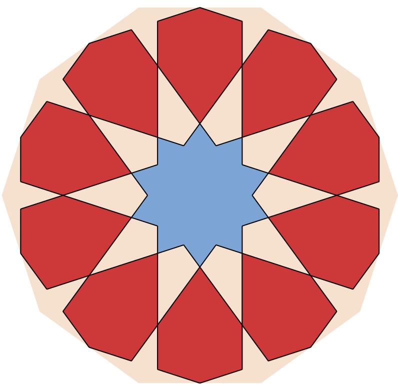


**A simple decagonal star**

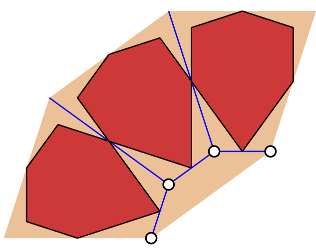
## Working with structures

In several Kukeldash Madrasah style patterns, it is possible to distinguish fragments that form a specific structure, i.e., a regular decagon, two overlapping decagons, a D-hexagon, etc. We will discuss some of these structures. While analyzing old patterns, we may often discover more structural elements. In the next series of drawings, we show only a very few of them.

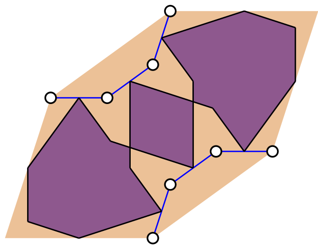
Fig. 8. Patterns for *D.b* structures



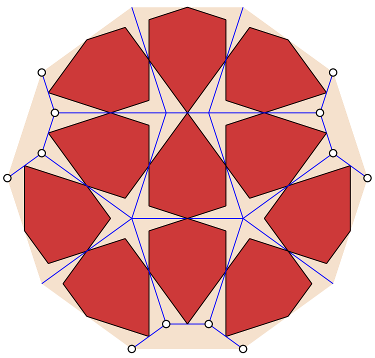
**S1. Large rosette** – this is the most obvious way of filling a decagon with the pattern (ten mirror lines).



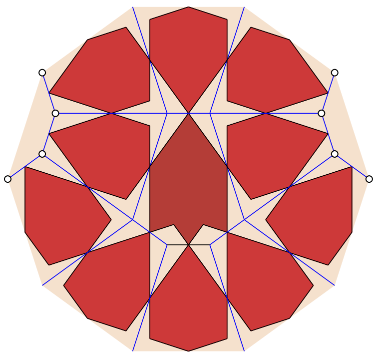
**S2. D-hexagon** – it can be part of a rosette or an independent element (one mirror line).



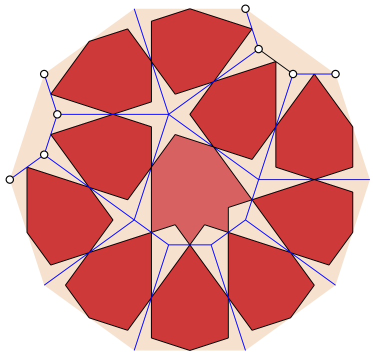
**S3. D-hexagon** with two gaps on sides. It is often seen structure in Maghrebi designs (two mirror lines).



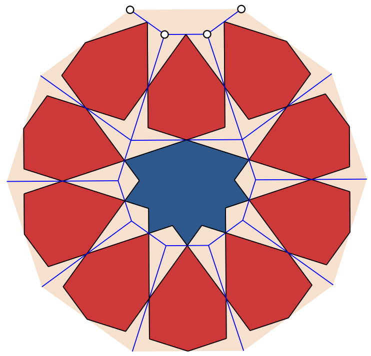
**S4. Alternative fill of a decagon** It needs three instances of petal 2 to fill gaps on the edge of the decagon (one mirror line)



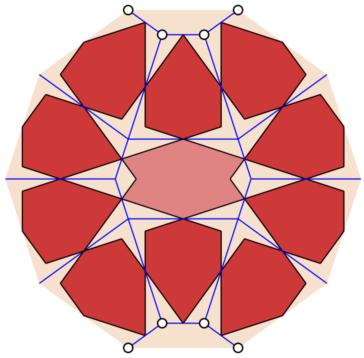
**S5. Another way to fill a decagon**. This time we need two instances of petal 2 to fill gaps on the edge (one mirror line)



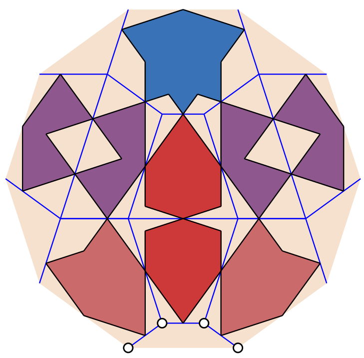
**S6. Yet another way to fill a decagon**. We need two instances of petal 2 to fill gaps on the edge (one mirror line through the decagon vertices).



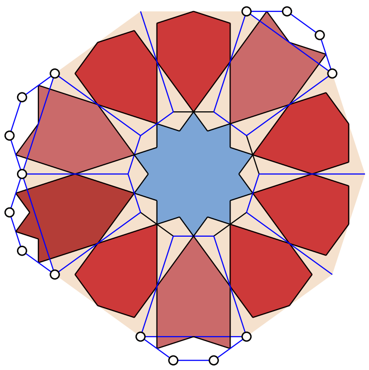
**S7. Yet another way to fill a decagon**. One mirror line



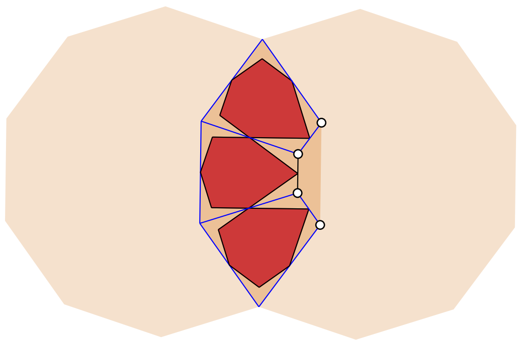
**S8. The only shape with two mirror lines**.



**S9. A rather fancy way to fill a decagon**. The four empty triangles can be covered by petals 3 (one mirror line).



**S10.** Depending on our need, any petal in S1 can be replaced by petals 2 or 3. The same applies to shapes S2-S8. Any petal 1 with a wide angle facing outside can be replaced by petals 2 or 3 if there is such a need.



**Two overlapping decagons (same as S3)**

The common part of two overlapping decagons can be filled with the pattern as it is shown here. The remaining parts can be filled with a pattern copied from S4 to S8. This situation frequently occurs in many Persian designs.

We can invent more ways to cover a regular decagon with petals shown in this paper. We can also cover with patterns some other polygons. This can be done depending on the needs of a particular design.

## Reconstructing the original Kukeldash Madrasah doors pattern

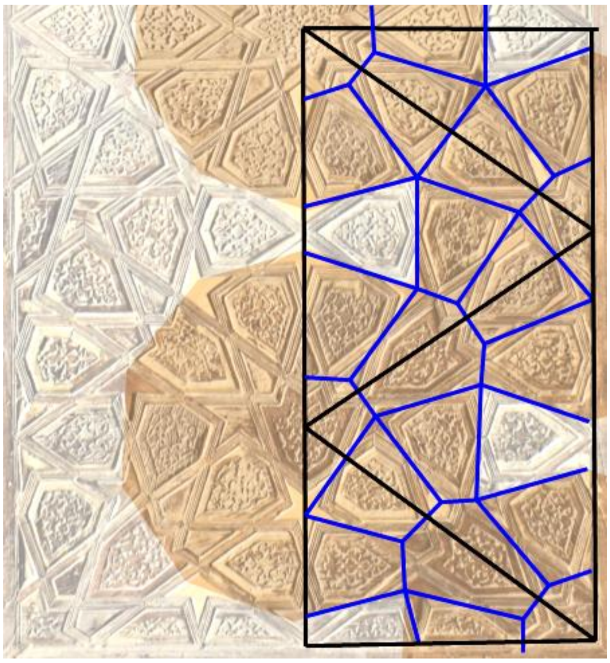


Fig. 9. The bottom half of the Kukeldash Madrasah doors

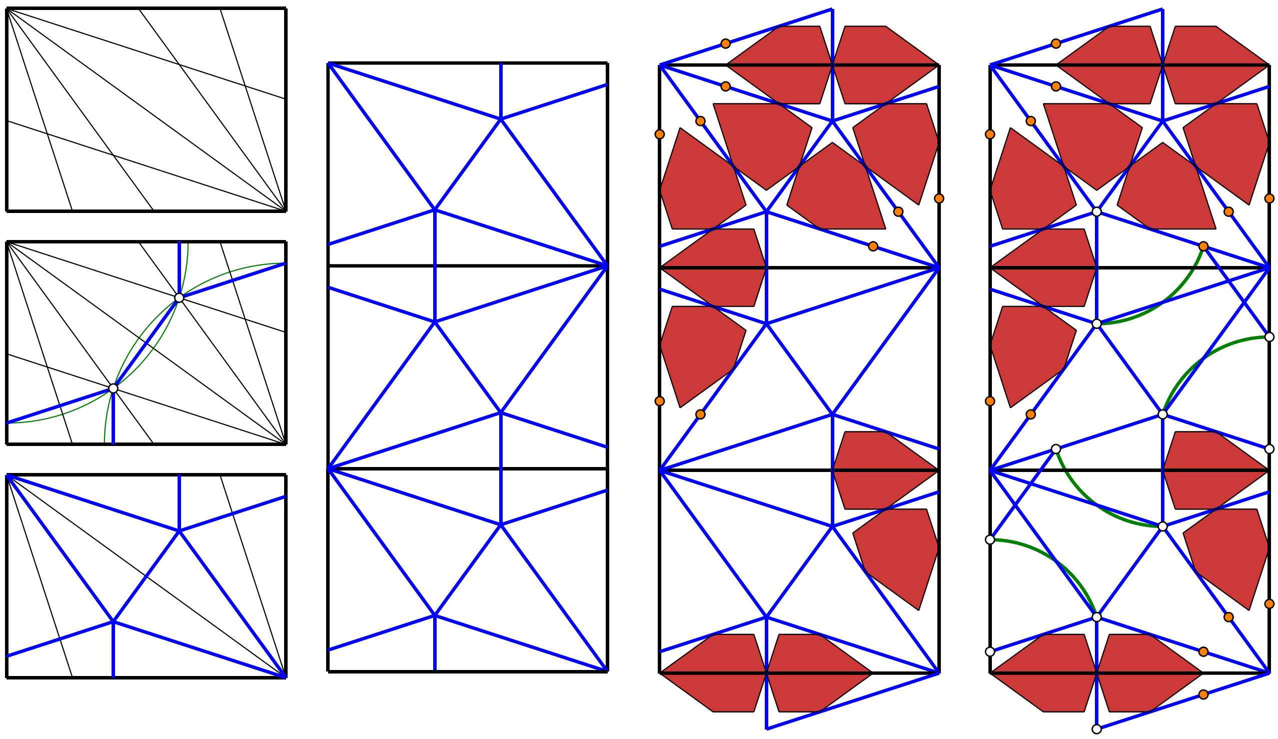
The hand drawing on a photo of the pattern shows four tangent decagons. The top one contains the central rosette of the pattern. The bottom one contains the corner rosette. The two remaining decagons were filled with a pattern, as it was shown in structures S6 and S10.

The black slant lines show how the contour and all decagons can be constructed.

There are many possible variations of this pattern. Each of them can be created by using different ways to fill each of these four decagons. We will show some of them only.

The next few drawings show a complete reconstruction of the pattern.

Fig. 10. Step-by-step reconstruction of the Kukeldash Madrasah pattern

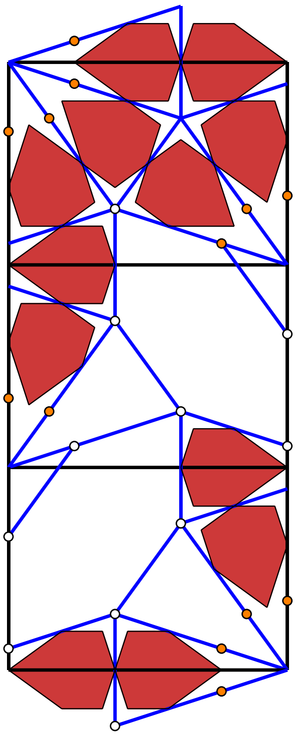


Three steps for creating the top part of the contour and tessellation.

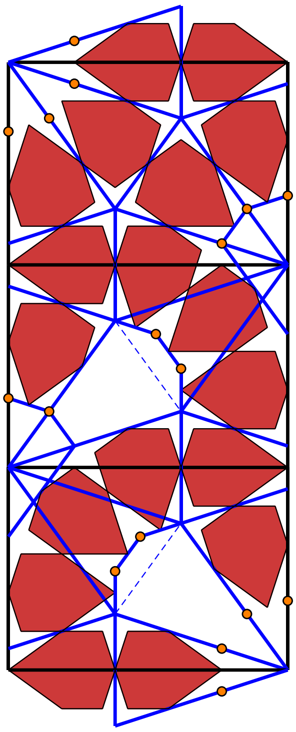
A temporary alternative tessellation. We will have to make some adjustments later.

Filling tessellation with petal 1.

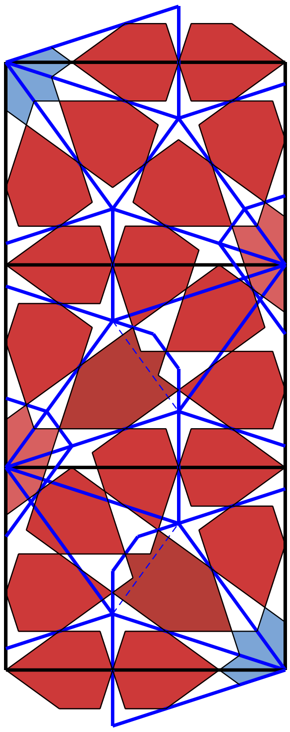
Creation of trapeziums that will allow us to develop structures S6 and S10.



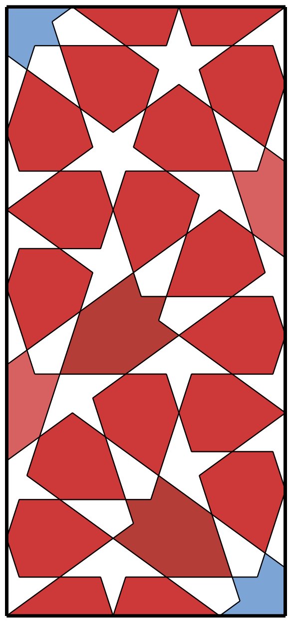
Modified tessellation.



Filling appropriate tiles with petal 1.

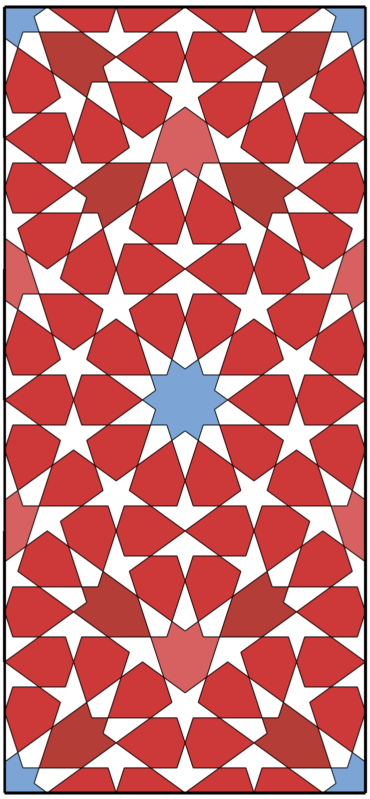


Adding missing petals 2 and 3.

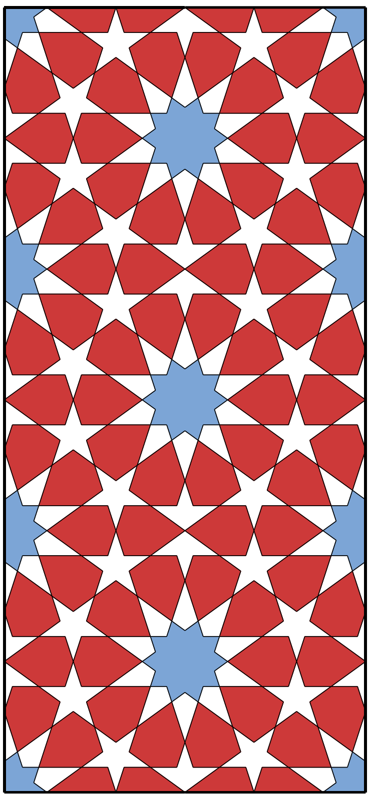


Quarter of the pattern from Kukeldash Madrasah main gate.

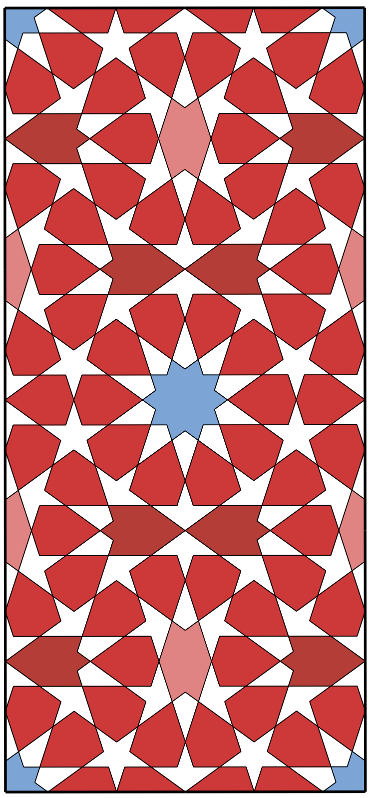
Below we will show a complete Kukeldash Madrasah pattern and two of its variants.



The original pattern from the Kukeldash madrasah



A very classic and the most popular decagonal pattern

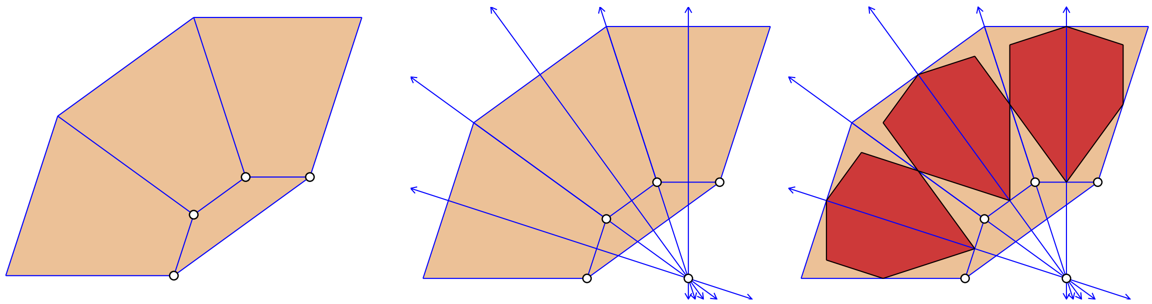


Another variation of the original pattern

## On tight packing *D*-hexagons

In the previous example, we used tangent decagons to construct a pattern. We were able to make a successful design and some variations of it. But this method is still very limited. While analyzing some old geometric constructions, we may find a few other ways to create Kukeldash style designs using alternative tessellation. One of them is the method of *tight-packed D-hexagons*. Let us start by analyzing a D-hexagon. We will call them *decagonal hexagons* or *D-hexagons* as they are built in decagonal geometry and are still hexagons. Below we show how a D-hexagon can be inflated, i.e., split into smaller shapes.

Fig. 11. Basic features of inflated D-hexagon

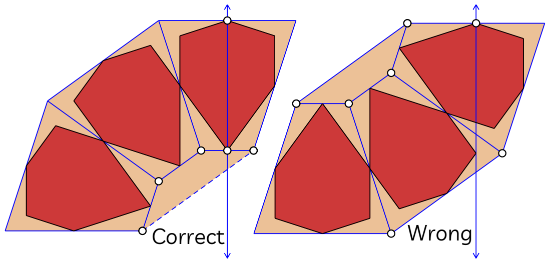


**Inflated *D*-hexagon**

This hexagon is part of many designs with decagonal tessellations. It cannot be used for the Kukeldash Madrasah style using the traditional approach. However, as we have noticed before, it may contain three petals 1 leaving a small gap on one of the edges. We can attach this gap to a neighboring tile and insert in this space one of the petals 2 or 3.

**Possible section lines of an inflated D-hexagon**

The dashed lines show where a contour edge can cut a *D*-hexagon. Each line is a symmetry line for one of the shapes inserted into the hexagon. This is the only way how we are allowed to cut a tile or a pattern shape by the contour of the whole design. Thus while designing patterns with *D*-hexagons, we must be sure that we correctly cut the hexagon.




The method of tight-packed hexagons requires constructing a set of identical tangent D-hexagons so that gaps between them will be smaller than a single D-hexagon. Then we can fill with a pattern the hexagons that are cut by the edges of the contour, including places where we expect to have large rosettes. Finally, we add petals into the remaining space. In this last step, we still may have some variations.

Note, the process of designing a pattern with tight packing hexagons has two moments where we can make some variations. One is the creation of D-hexagons, and the other is when we decide how to place gaps on a D-hexagon.

The largest difficulty in such a design is still the same as with other methods – creating a proper contour and creating the first tile. In many designs, the first tile is a quarter of a decagon.

### Reconstructing pattern from David's Collection

David's Collection is a museum in Copenhagen in, Denmark (see https://www.davidmus.dk/en/). The museum contains several interesting artifacts from Iran and other places. In this museum, we can find two wooden doors of Iranian origin. One of them has a pattern in Kukeldash Madrasah style. This will be our object to experiment with it in this chapter. Note - in Abdulloxon Madrasah in Bukhara, Uzbekistan, we can find a pattern created using the same tessellation of tight-packed hexagons.

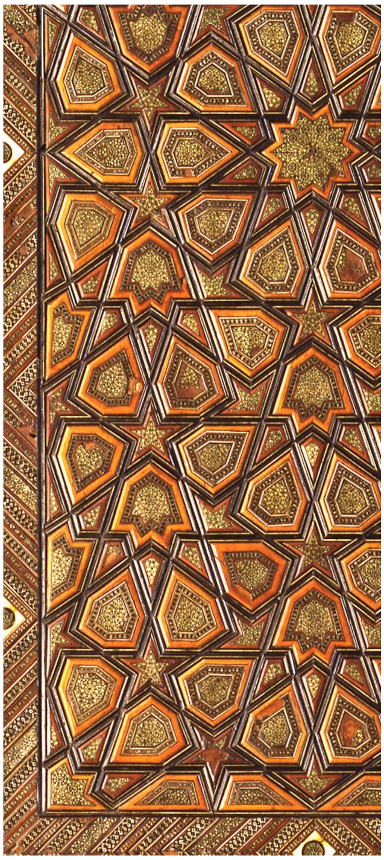


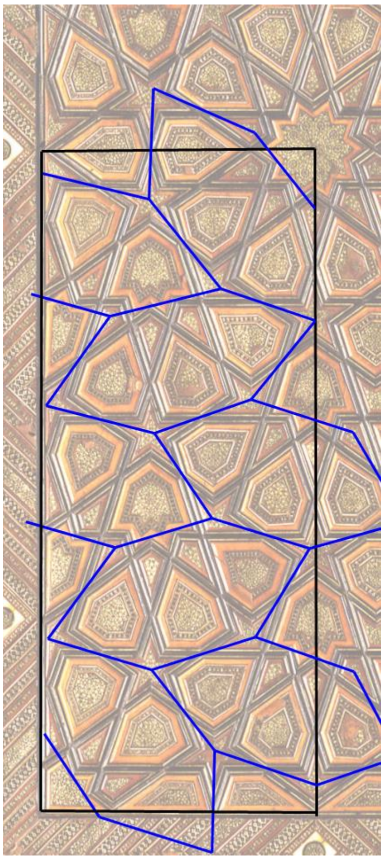
**Fig. 12. Pattern from the doors in David's Collection Museum**

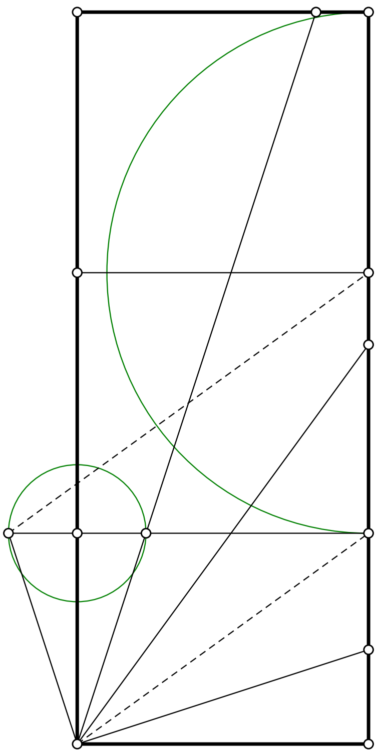
**Left:** the left-bottom corner of the door pattern. One can easily notice triplets of shapes forming a series of *D*-hexagons.

**Right**: the same pattern with a hand-drawn tessellation of thigh-packed decagonal hexagons.

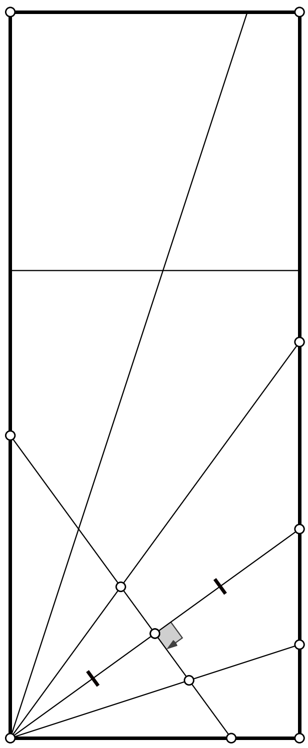
We will start by constructing this rather unusual contour. Then we will construct the quarter of a decagon in the left bottom corner. Then we will add pattern petals to the *D*-hexagons on the right and left edges. Finally, we will add pattern petals to the remaining hexagons.

**NOTE:** some parts of the hexagons on the left edge should be left empty until the very last step of construction.

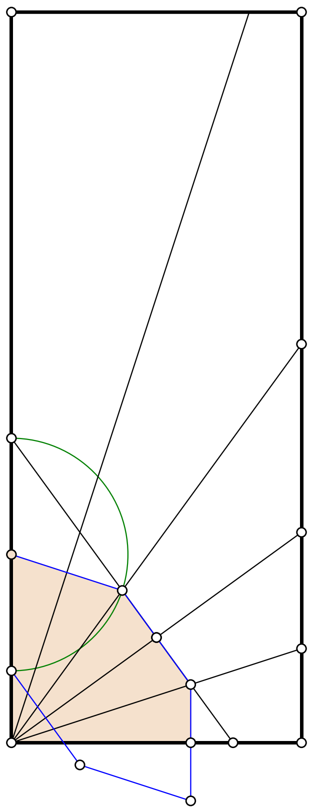




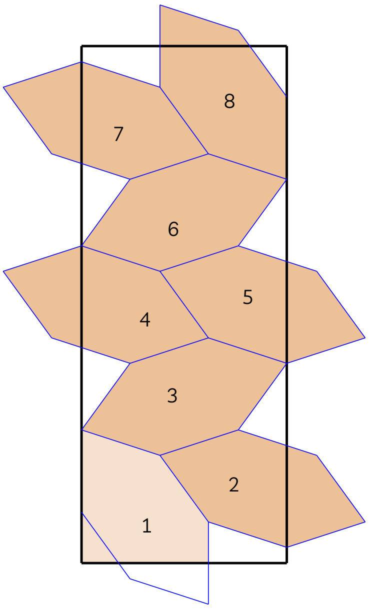
Construction of the contour: start from the bottom edge and create the three rectangles. The dashed lines are parallel.



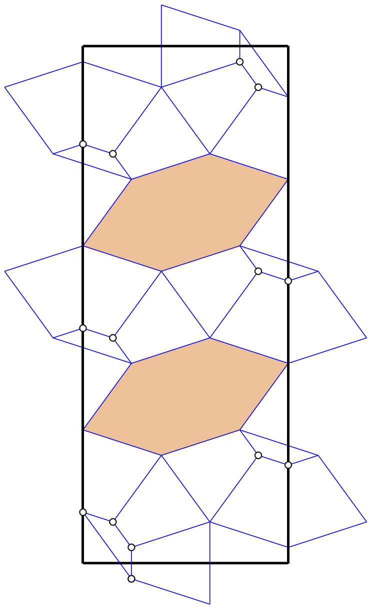
Location of the first D-hexagon and its edge length



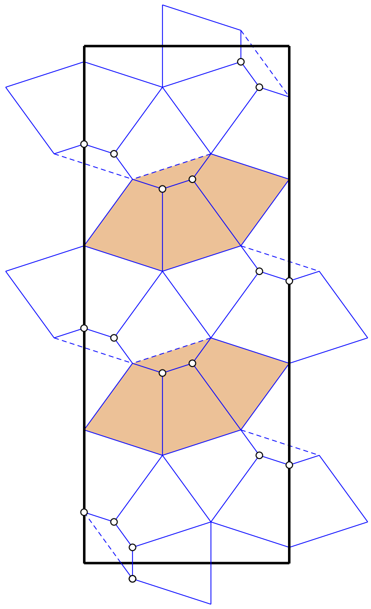
Here we show how the first D-hexagon will be located. Each next hexagon will be a copy of the first one.



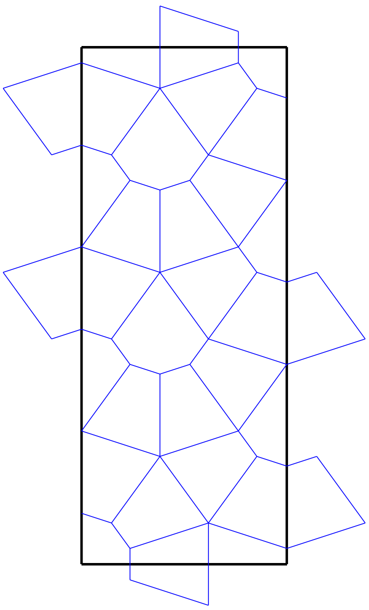

Having the first D-hexagon, we stack the copies of it according to the numbers given.



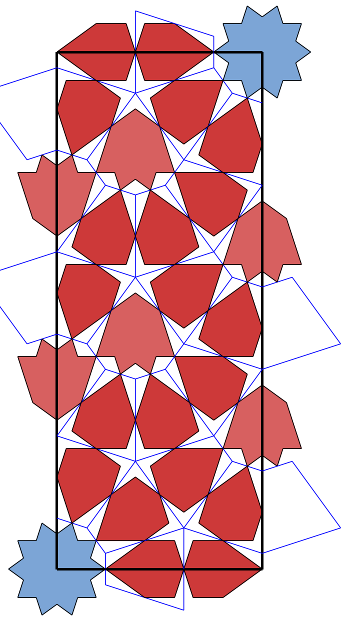
Here we fill hexagons on edges with trapeziums for petal 1. Here we do not have a choice. We have to follow the rules. We are left with two hexagons where we have some choice on how to split them.



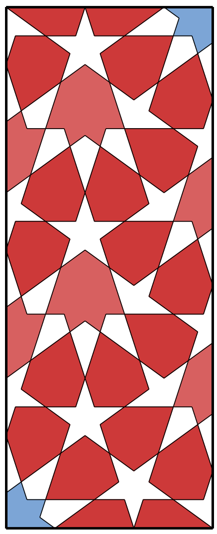
The dashed lines show which edges of the blue tessellation should be removed.



Final tessellation for the alternative approach



Final pattern with its tessellation



A quarter of the pattern from the doors in David's Collection Museum

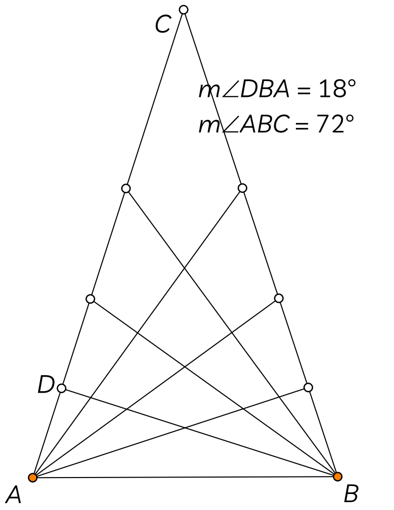
The pattern from David's Collection Museum is very simple. Several other similar designs can be created using the tight-packed hexagons method. Some of them can be very large and very complex. However, this method and the tangent decagons approach are still very limited. Therefore we have to look into a more flexible way of approaching decagonal patterns with *D.b* tessellations.

## On structural designs

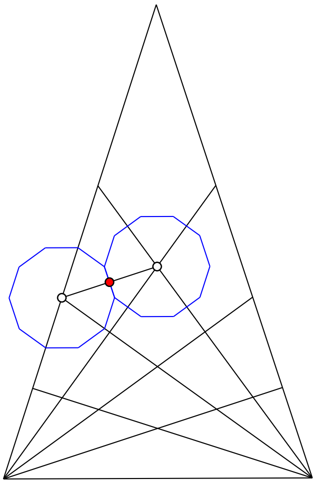
While developing complex patterns, it is convenient to construct an overall structure of the pattern, fill it with a tessellation, and finally add pattern elements. In Persian geometric art, there are many such designs. Here we will show only one of them.

### Example: A Zizi decagonal medallion

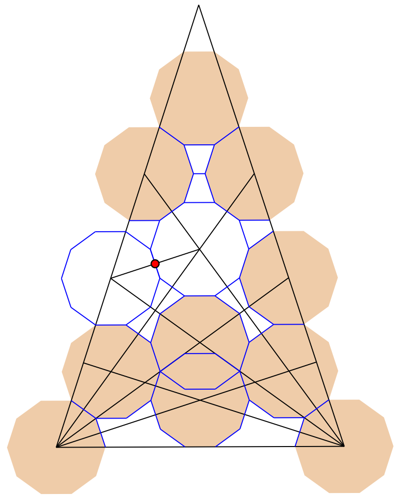
The objective of this example is to create a large medallion fitted into a regular decagon. We will construct only 1/10 of the whole design and then make ten copies of it. Here we show how one can proceed. Note – this is only one of many possible designs.



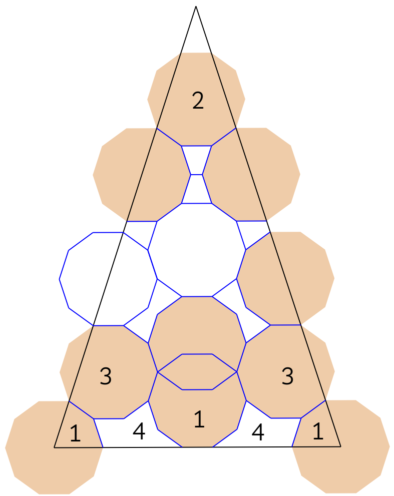

A quarter of a decagon and a division of its angles forms a grid that we may use to design a series of tangent and/or overlapping decagons.



One of many possible ways to create the first decagon. All other decagons should be copies of this one.

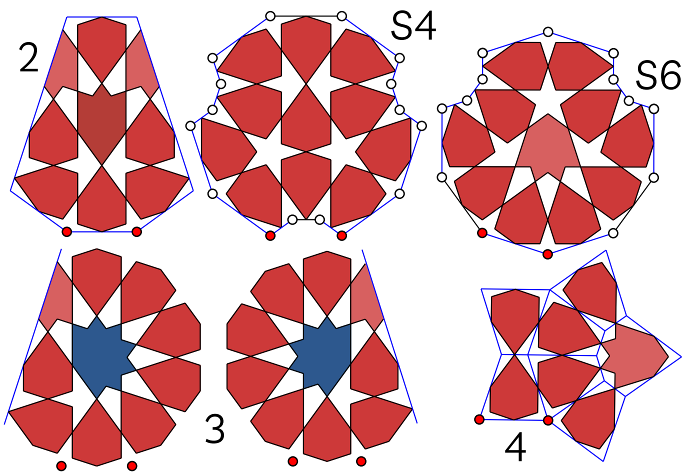


An example of how we can build a tessellation of tangent and overlapping decagons. There are many other ways.




#### Tessellation of tangent and overlapping decagons

The challenge are polygons 2 and 3. Number 1 shows places where we want to have proper rosettes with all identical petals.

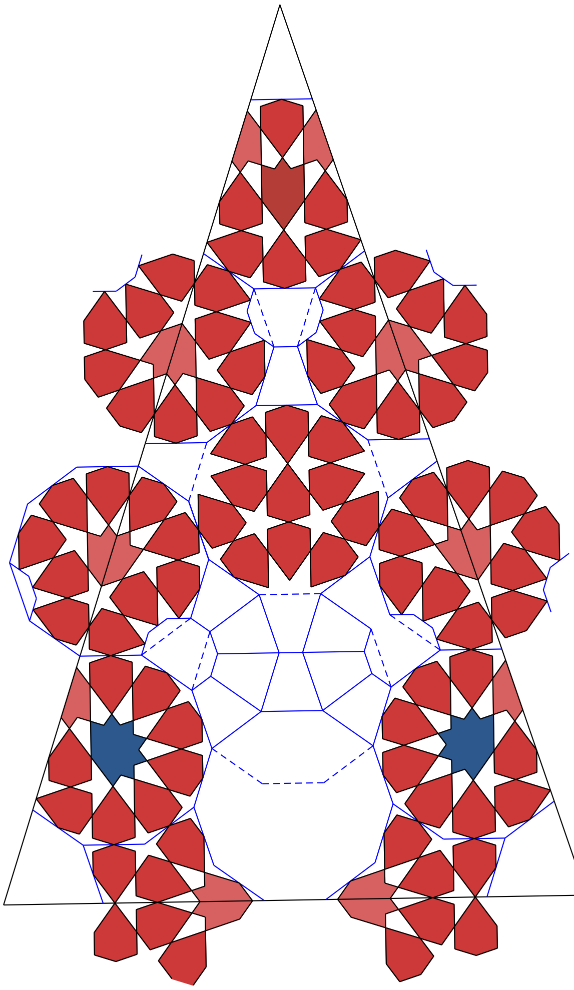



#### A proposition of fills for decagons in this design

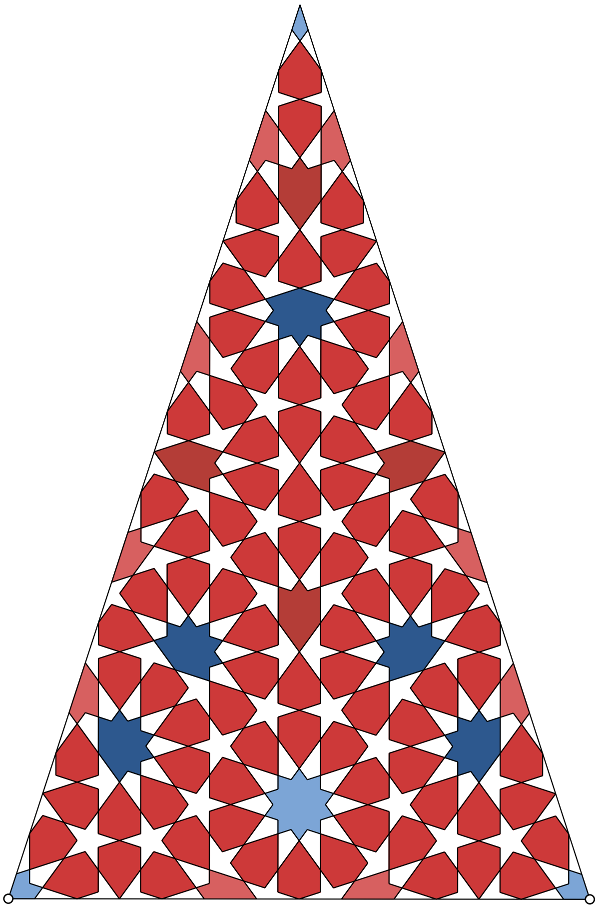
Polygon 2 can be replaced by discussed earlier S9.

Polygon 3 in two versions – for the left and right edge of the triangle. Polygon S4 was discussed before and can be replaced by any of S5, S6, S7, S8, or even S9. S6 we need for the right and left edges of the triangle.

Now we can develop a complete tessellation for the triangle and then fill it with a pattern.



This drawing shows how the proposed fills fit into the whole design. The dashed lines should be removed. The remaining empty polygons should be filled with appropriate pattern shapes.

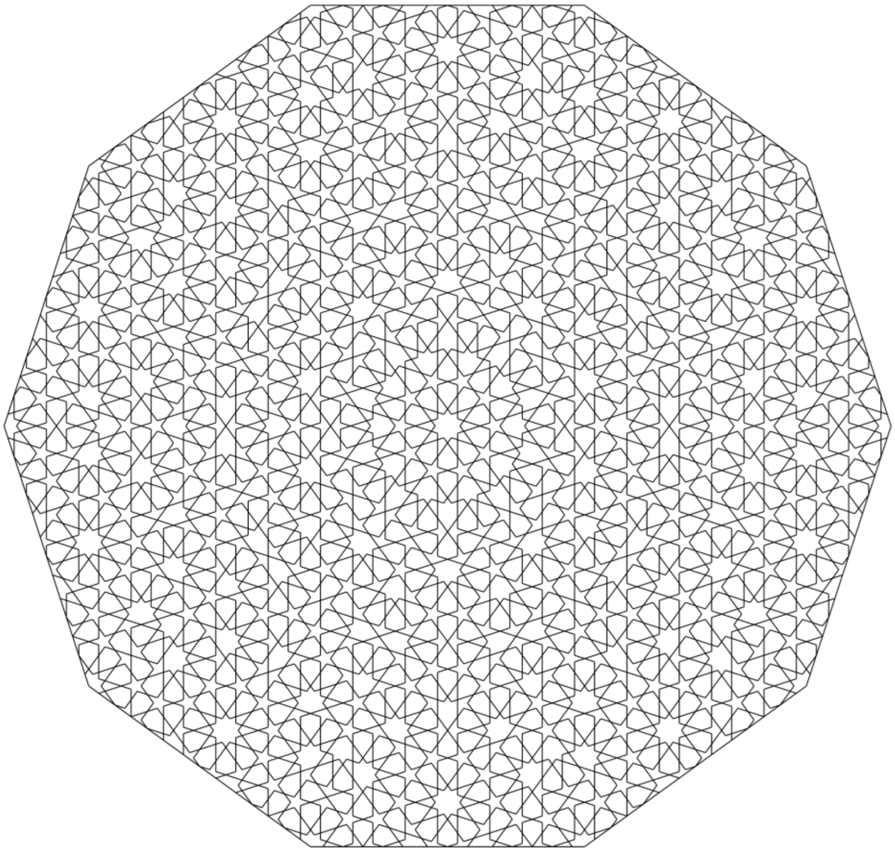


The drawing shows a complete design of 1/10 of a pattern to cover a regular decagon.

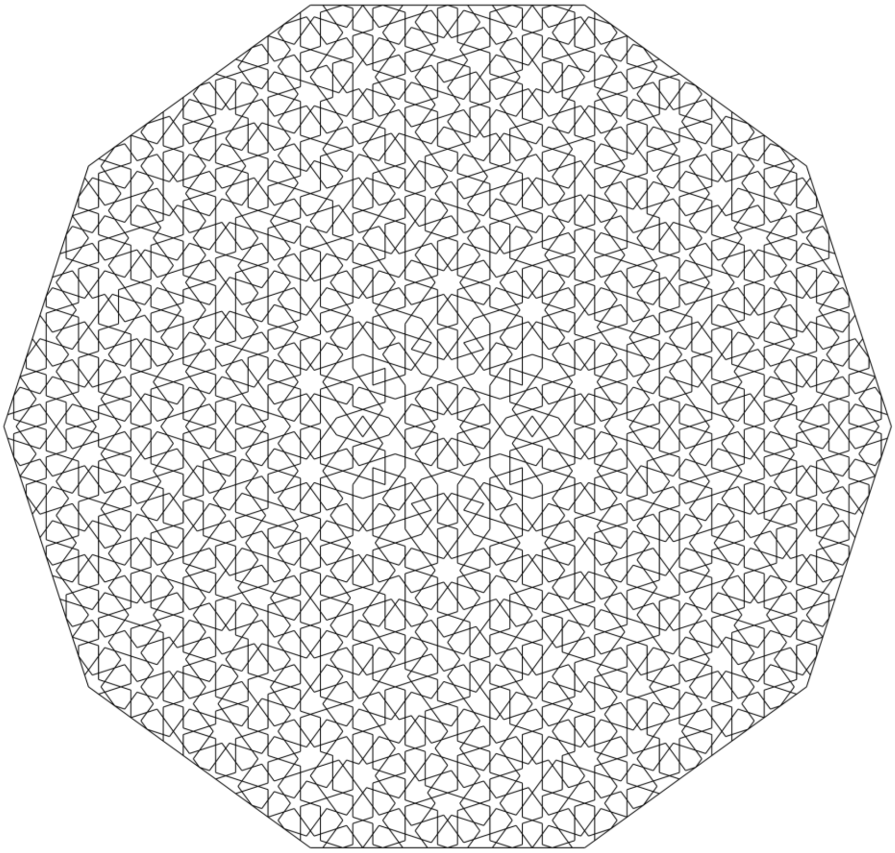
The tangent and overlapping decagons method can be quite successful while designing large panels with multiple shapes. In the above design, we used a few known to us shapes. However, we could make this design even more complex. We could also use a few slightly different color versions of each of the shapes. This way, we could have a more interesting design. However, this will not change the geometry of its structure. The geometric structure will be exactly the same.

## Conclusions

The concepts presented in this paper were invented by the author in the spring of 2022 while teaching an online course for Istanbul Design Center students. None of them has been published in any known paper or book. None of them was presented or mentioned to the author by another person. The geometry of alternative tessellations is a concept that still needs more research and experiments. This paper presented only a few examples from a huge collection of patterns in the Kukeldash Madrasah style. Some of them are very interesting. Some are quite difficult to design using alternative tessellations. Hopefully, a more complete theory of designing patterns with alternative tessellations will be a topic of another paper or a book.



The Zizi decagonal medallion (edges only) and its modification below



**FINAL COMMENTS:** All drawings and geometric constructions in this paper were created using the free version of Geometer's Sketchpad.

## References


[1] Majewski. M. (2019a). *Practical Geometric Pattern Design – decagonal patterns in Islamic art* (part 1). Istanbul: Istanbul Design Publishing.

[2] Majewski. M. (2019b). *Practical Geometric Pattern Design – decagonal patterns in Islamic art* (part 2). Istanbul: Istanbul Design Publishing.

[3] Majewski. M. (2020). *Practical Geometric Pattern Design: Geometric Patterns from Islamic Art*. Kindle Direct, Independently published (February 10, 2020).

[4] Majewski. M. (2020). *Understanding Geometric Pattern and its Geometry* (*part* 1), eJMT, vol. 14, Nr 2, pages 87-106

[5] Majewski. M. (2020). *Understanding Geometric Pattern and its Geometry* (*part* 2) – Decagonal Diversity, eJMT, vol. 14, Nr 3, pages 147-161.

[6] Majewski. M. (2021). *Understanding Geometric Pattern and its Geometry* (*part* 6) – Using Geometer's Sketchpad for designing sizeable geometric projects, ATCM 2021, (http://atcm.mathandtech.org/EP2021/invited/21875.pdf).